Reconstruction Problems in Computer Vision

Applications and Algebra

Felix Rydell

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Agenda

The aim of this presentation is to introduce computer vision with applications, motivate and define basic concepts of algebraic geometry, and finally I will talk about a current project.

This talk is divided into three parts:

1. Computer vision, applications and reconstruction problems,
2. Introduction to algebraic geometry,
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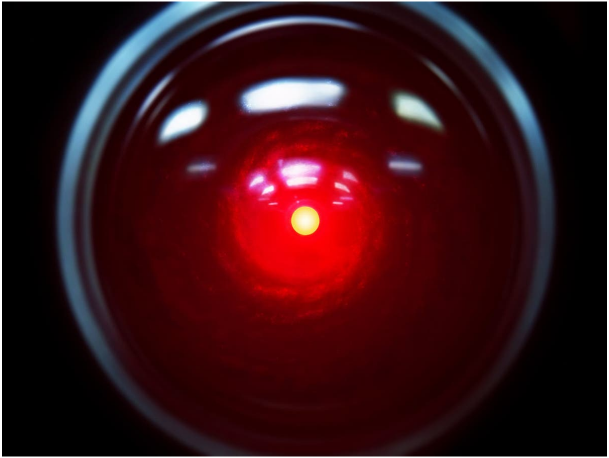

Figure: Hal 9000, the artificial intelligence from *2001: A Space Oddysey (1968)*.

What Is Computer Vision?

- *Computer vision* is an umbrella term. It is essentially the study of how computers can learn information from images (and videos.) This is the *perceptual* component of intelligence. It seeks to understand and automate tasks that the human visual system can do.

- Examples of tasks in computer vision:
  - Reading hand-written text, number plates (convolutional neural networks,)
  - Self-driving cars,
  - Deep-fakes, making faces older or younger in films,
  - Surveillance, monitoring pools for drowing people,
  - Object recognition for automated check out lanes,
  - Increasing resolution of pictures and denoising (auto-encoders,)
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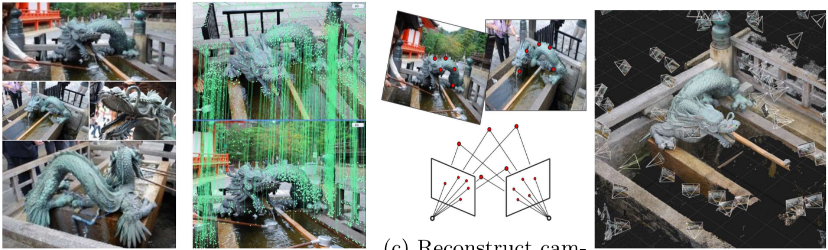

(a) Input images (b) Image matching (c) Reconstruct cameras and 3D points (d) Output

Figure: The reconstruction pipeline.

Reconstructing 3D models

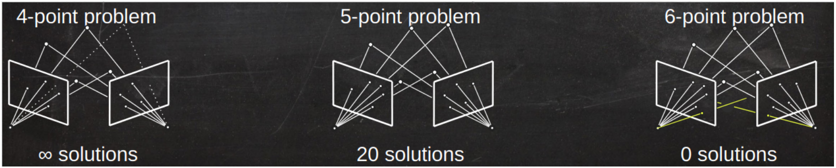

Figure: Different algebraic reconstruction problems. The number of complex solutions are listed for generic positions (up to rotation and translation.)

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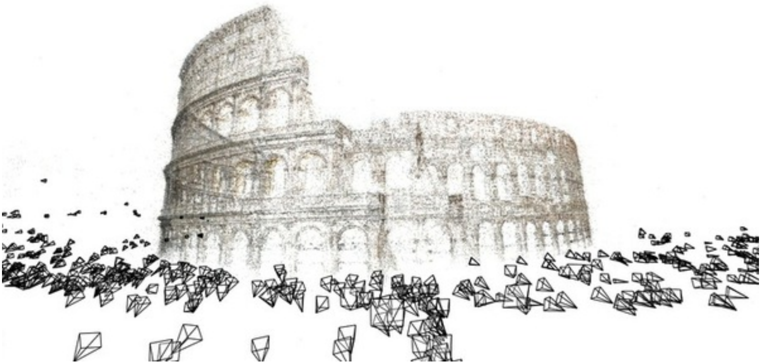

Figure: Point reconstruction of the colloseum using pictures found on the internet taken by tourists' smartphones. Also the camera position (and their directions) have been recovered. Taken from *Reconstructing Rome,* by Agarwal, Furukawa et. al.

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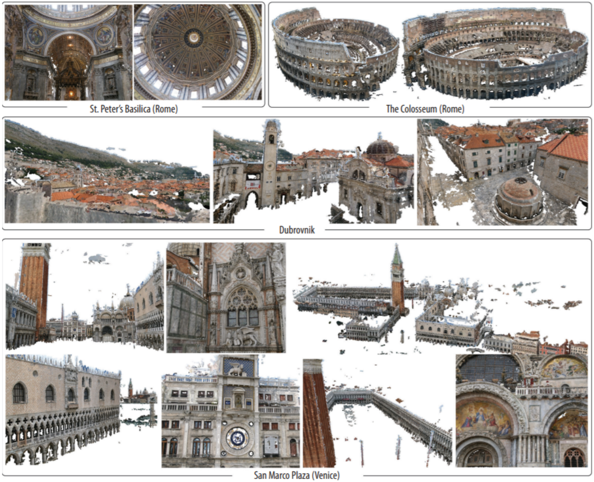

Figure: Finished 3D models. The pictures used were taken by tourists. Taken from *Reconstructing Rome*, by Agarwal, Furukawa et. al.

Types of Cameras

- There are many types of cameras, and in some cases the type really matters in applications:
  - *Pinhole camera:* The "easiest" kind, takes the whole picture at once. Well-understood from a mathematical perspective,
  - *Rolling shutter camera:* Common in smartphones, scans each line of the image one by one,
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Types of Cameras

Figure: Picture taken with a rolling shutter camera shows bent rotor blades. In our 3D reconstruction, we want the rotor blades to be straight.

Part 2 – Algebraic Geometry

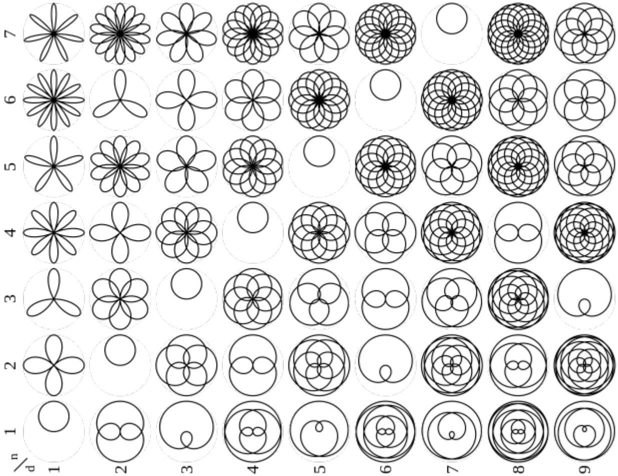

Figure: The family of *rose curves*.

Motivation

- *Algebraic geometry* is the study of solutions to polynomial equations. For example,

$$V = \{(x, y) \in K^2 : x^2 + y^2 = 1\},$$

where K is a field (usually $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$.)

- Observe that $\mathbb{C}$ is an algebraically closed field (due to the fundamental theorem of algebra,) which means that its geometry is "nice" and "simple." The real numbers $\mathbb{R}$ model the real world and computers can symbolically work with rational numbers $\mathbb{Q}$.

- This field can be thought of as a generalization of linear algebra, which is the study of the solution set of linear equations.

- The world can be understood through polynomials! In particular, real valued functions can in a broad sense be approximated by polynomials, via Taylor approximation or Stone-Weierstrass theorem.

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Geometry vs. Algebra

- Consider the ring of polynomials $K[x_1, \ldots, x_n]$ ($K = \mathbb{Q}, \mathbb{R}, \mathbb{C}$.) An ideal $\mathcal{I}$ of this ring is a subring such that $f \cdot \mathcal{I} \subseteq \mathcal{I}$ for any polynomial $f \in K[x_1, \ldots, x_n]$. The ideal generated by polynomials $f_1, \ldots, f_k$ is

$$\langle f_1, \ldots, f_k \rangle := \left\{ \sum_{i=1}^{k} a_i f_i : a_i \in K[x_1, \ldots, x_n] \right\}.$$

- An *algebraic set* (variety) $\mathcal{V}(\mathcal{I})$ is the set of points in K^n for which all polynomials in $\mathcal{I}$ vanish. **Key idea:** If $f_1, \ldots, f_k$ generate $\mathcal{I}$, then

$$\mathcal{V}(\mathcal{I}) = \{x \in K^n : f_1(x) = \cdots = f_n(x) = 0\}.$$

A computer understands algebraic geometry (an ideal) as a set of finitely many generators:

Theorem (Hilbert's Basis Theorem)

If $K = \mathbb{Q}, \mathbb{R}, \mathbb{C}$ then any ideal of $K[x_1, \ldots, x_n]$ is finitely generatored.

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- An *algebraic set* (variety) $\mathcal{V}(\mathcal{I})$ is the set of points in K^n for which all polynomials in $\mathcal{I}$ vanish. **Key idea:** If $f_1, \ldots, f_k$ generate $\mathcal{I}$, then

$$\mathcal{V}(\mathcal{I}) = \{x \in K^n : f_1(x) = \cdots = f_n(x) = 0\}.$$

A computer understands algebraic geometry (an ideal) as a set of finitely many generators:

Theorem (Hilbert's Basis Theorem)

If $K = \mathbb{Q}, \mathbb{R}, \mathbb{C}$ then any ideal of $K[x_1, \ldots, x_n]$ is finitely generatored.

Geometry vs. Algebra

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Projective Geometry

- Projective geometry was "first studied" by painters long ago who wanted to draw proportions of structures and buildings in a realistic way.
- **Key idea:** Compact sets have nice properties that are nice to work with. Sets as $\mathbb{R}^n$ and $\mathbb{C}^n$ are non-compact, but this can be fixed via *projectivization.*
- For a field K, we define the projective space

$$\mathbb{P}_K^{n-1} := \{L \subseteq K^n : L \text{ is } 1-\dim \text{ linear subspace}\} = (K^n \setminus \{0\})/\sim,$$

where $x, y \in K^n$ are related by $\sim$ if they differ by a non-zero constant. Its elements are written $(x_1 : \cdots : x_n)$, for some $(x_1, \ldots, x_n) \in K^n$ that spans the line.

- **Key idea:** Just as algebraic geometry becomes easier over $\mathbb{C}$ instead of $\mathbb{R}$, it also becomes easier in projective space compared to the usual *affine space.*
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Grassmannians

- One example of a common algebraic construction is the *Grassmanian*, defined as

$$\mathrm{Gr}(k, \mathbb{R}^n) := \{L \subseteq \mathbb{R}^n : L \text{ linear space of dim } k\}.$$

In terms of projective space, we have a natural correspondence

$$\mathrm{Gr}(k-1, \mathbb{P}^{n-1}_{\mathbb{R}}) \cong \mathrm{Gr}(k, \mathbb{R}^n).$$

- **Key idea:** These can be viewed as algebraic sets. This allows us to work efficiently with linear spaces, viewing them as points.

- For example, $\mathrm{Gr}(1, \mathbb{P}^3) \cong \mathrm{Gr}(2, \mathbb{R}^4)$ lie in $\mathbb{P}^5$ by the following *Plücker embedding* sending the space spanned by a, b, represented as the matrix

$$\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{bmatrix}$$

to the vector of the six 2×2 minors of this matrix in $\mathbb{P}^5$. This is a *natural* bijection.

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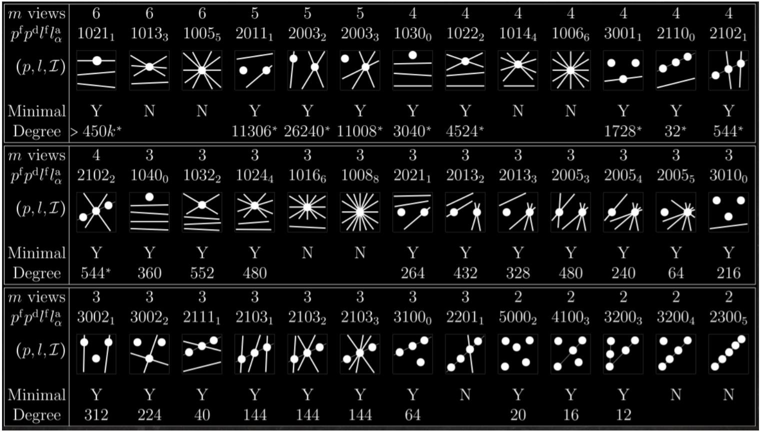

m views	6	6	6	5	5	5	4	4	4	4	4	4	4
$p^{\mathrm{f}}p^{\mathrm{d}}l^{\mathrm{f}}l^{\mathrm{a}}_{\alpha}$	1021_1	1013_3	1005_5	2011_1	2003_2	2003_3	1030_0	1022_2	1014_4	1006_6	3001_1	2110_0	2102_1
$(p, l, \mathcal{I})$													
Minimal	Y	N	N	Y	Y	Y	Y	Y	N	N	Y	Y	Y
Degree	$> 450k^*$			11306*	26240*	11008*	3040*	4524*			1728*	32*	544*

m views	4	3	3	3	3	3	3	3	3	3	3	3	3
$p^{\mathrm{f}}p^{\mathrm{d}}l^{\mathrm{f}}l^{\mathrm{a}}_{\alpha}$	2102_2	1040_0	1032_2	1024_4	1016_6	1008_8	2021_1	2013_2	2013_3	2005_3	2005_4	2005_5	3010_0
$(p, l, \mathcal{I})$													
Minimal	Y	Y	Y	Y	N	N	Y	Y	Y	Y	Y	Y	Y
Degree	544*	360	552	480			264	432	328	480	240	64	216

m views	3	3	3	3	3	3	3	3	2	2	2	2	2
$p^{\mathrm{f}}p^{\mathrm{d}}l^{\mathrm{f}}l^{\mathrm{a}}_{\alpha}$	3002_1	3002_2	2111_1	2103_1	2103_2	2103_3	3100_0	2201_1	5000_2	4100_3	3200_3	3200_4	2300_5
$(p, l, \mathcal{I})$													
Minimal	Y	Y	Y	Y	Y	Y	Y	N	Y	Y	Y	N	N
Degree	312	224	40	144	144	144	64		20	16	12		

Figure: Different algebraic reconstruction problems with finitely many solutions. Taken from *PLMP - Point-Line Minimal Problems in Complete Multi-View Visibility*, by Kohn et. al.

Modelling Cameras Mathematically

- A camera is modelled mathematically as a 3×4 matrix C of rank 3, projecting points (or lines) in $\mathbb{P}^3$ to the camera image plane $\mathbb{P}^2$. The *focus* (center) of the camera is equal to the kernel of C.

- Let $C^{(1)}, \dots, C^{(m)}$ be m camera matrices with different foci. We define the *joint camera map*

$$\Phi_C : \mathbb{P}^3 \dashrightarrow (\mathbb{P}^2)^m, X \mapsto (C^{(1)}X, \dots, C^{(m)}X),$$

sending a point in 3-space to its point images in the m cameras.

- Similarly, we have the alternative setting

$$\Upsilon_C : \mathrm{Gr}(1, \mathbb{P}^3) \dashrightarrow \mathrm{Gr}(1, \mathbb{P}^2)^m, L \mapsto (C^{(1)}L, \dots, C^{(m)}L),$$

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Multi-View Varieties

- The reconstruction problem becomes easier when we know the camera specifications, i.e. we know the matrices $C^{(1)}, \ldots, C^{(m)}$. Reconstructing in this case becomes an exercise in linear algebra, after the image points (or lines) have been fitted into the multi-view variety.

- **Key idea:** It's difficult for a computer to work with the *joint images* $\Phi_C(\mathbb{P}^3)$ and $\Upsilon_C(\mathrm{Gr}(1, \mathbb{P}^3))$. Instead, we consider the *Zariski closure* of these images, written $\mathcal{M}_C$ in the point case and $\mathcal{L}_C$ in the line case. These are called *multi-view varieties.* The Zariski closure of a set A is the smallest algebraic set containing A.

- Any data comes with noise and we therefore fit them into the multi-view varieties by finding the closest point to it in Euclidean distance.

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Multi-View Varieties

- In the *generic case* (the cameras are chosen randomly for instance,) we can write the ideals defining the multi-view varieties. Geometrically, they are explained as follows:

Theorem

In the point case, the variety consists of the image points whose back-projected lines in $\mathbb{P}^3$ meet in a point. In the line case, the variety conists of the image lines whose back-projected planes in $\mathbb{P}^3$ meet in a line.

- Practicioners are interested in the robustness of the two approaches. There is numerical evidence that the line approach is more stable. We want algebraic evidence for this.

- In this direction, the *Euclidean distance degree* of a variety $\mathcal{V} \subseteq \mathbb{R}^n$ is the number of complex critical points of

$$\|v - x\|_2^2,$$

over $v \in \mathcal{V}$ and generic $x \in \mathbb{R}^n$ (randomly chosen.)

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Thank you for listening!

- A few relevant sources:
  - S. Agarwal, Y. Furukawa et.al. *Reconstructing Rome.*
  - R. Szeliski. *Computer Vision, Algorithms and Applications.*
  - M. Trager, M. Hebert, J. Ponce. *The joint image handbook.*
  - J. Kileel. *Algebraic Geometry for Computer Vision.*
  - K. Kohn et. al. *PLMP - Point-Line Minimal Problems in Complete Multi-View Visibility.*