Constraint Satisfaction Problems

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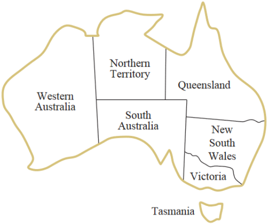
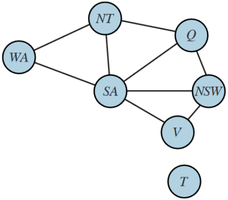
Example 1: Linear Equations

Does this system of linear equations modulo 2 have a solution?

$$
\begin{aligned}
x_1 + x_2 + x_5 &= 1 \pmod 2 \\
x_1 + x_3 + x_4 &= 1 \pmod 2 \\
x_2 + x_3 + x_5 &= 0 \pmod 2 \\
x_2 + x_3 + x_4 &= 1 \pmod 2
\end{aligned}
$$

Example 2: Coloring





Can we color all states of Australia with red, green, and blue so that no two neighboring states are assigned the same color?

Example 3: Sudoku

	1	2	3	4	5	6	7	8	9
A			3		2		6		
B	9			3		5			1
C			1	8		6	4		
D			8	1		2	9		
E	7								8
F			6	7		8	2		
G			2	6		9	5		
H	8			2		3			9
I			5		1		3		

Can we fill in the blanks with digits $1, 2, \ldots, 9$ so that no digit appears twice in any row, column or 3×3 box?

Common Framework – CSP

Constraint Satisfaction Problem (CSP)

INSTANCE: (V, D, C), where

- V is the set of variables,
- D is the set of values,
- C is the set of constraints.

QUESTION: Is there an assignment of values to the variables that satisfies all constraints?

Example 1: Linear Equations as CSP

Does this system of linear equations modulo 2 have a solution?

$$
\begin{aligned}
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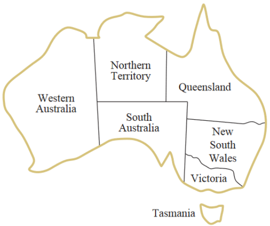
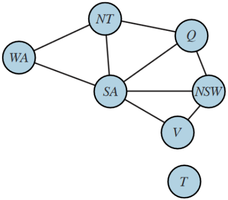
\end{aligned}
$$

Variables: $\{x_1, x_2, x_3, x_4, x_5\}$.
Values: $\{0, 1\}$.
Constraints are $x + y + z = 0 \text{mod} 2$, $x + y + z = 1 \text{mod} 2$.

Example 2: Coloring as CSP

Can we color all states of Australia with red, green, and blue so that no two neighboring states are assigned the same color?





Variables: $\{WA, NT, SA, NSW, V, SA, T\}$.

Values: {red, green, blue}.

Constraints: $WA \neq NT$, $WA \neq SA$, $NT \neq SA$, $\ldots$

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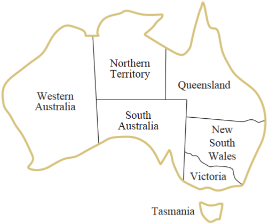

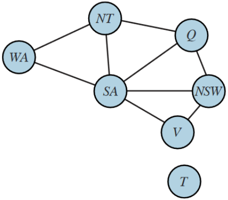

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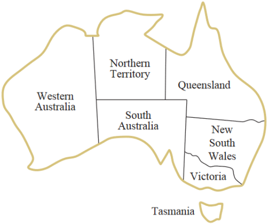

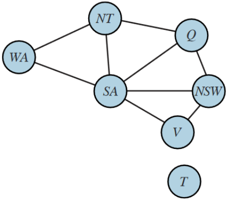

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Variables: `A1`, `A2`, ..., `I9` (81 in total).

Values: $1, 2, 3, 4, 5, 6, 7, 8, 9$.

Constraints are of two types: (i) $\texttt{B1} = 9, \ldots, \texttt{H9} = 9$ and (ii) $\texttt{AllDiff(A1}, \ldots, \texttt{A9)}$, $\texttt{AllDiff(A1}, \ldots, \texttt{I1)}$, $\texttt{AllDiff(A1}, \ldots, \texttt{C3)}$, ...

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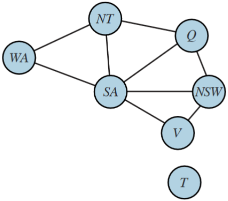
- 2-Coloring: yes! Use constraint propagation (next slide).
- 3-Coloring: probably not.
- CSP in general: probably not.
- probably stands for "unless P=NP" (more on that later).

Local Consistency: Example

Solving 2-Coloring with $(2, 3)$-consistency algorithm.



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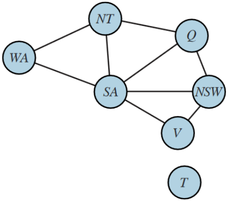

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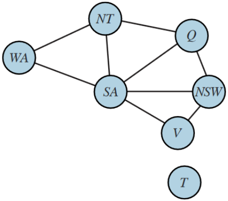

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WA	SA	V
0	0	0
0	0	1
0	1	0
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1	0	0
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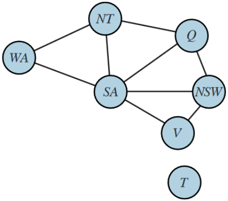

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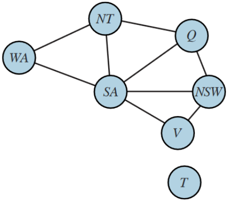

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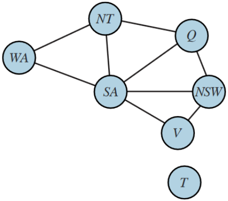

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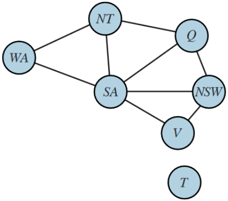

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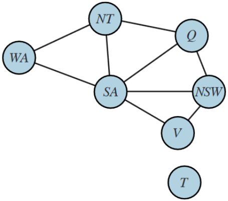

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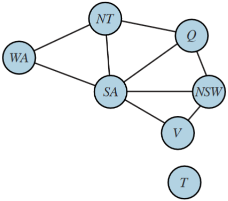

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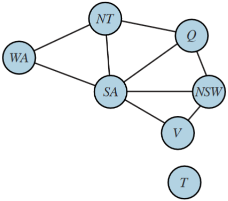

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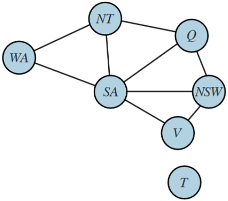

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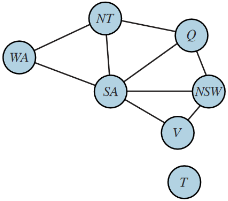

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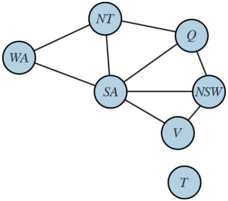

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- For 2-Coloring, $(2,3)$-consistency test is sufficient.
- Running time: $O(|V|^3)$.

Local Consistency

- Which CSPs are locally consistent?
- Restrict the set of values and the set of possible constraints.
- $\mathrm{CSP}(\{0,1\}; \neq)$ is $(2,3)$-consistent.
- $\mathrm{CSP}(\{0,1\}; x \oplus y \oplus z = 0, x \oplus y \oplus z = 1)$ is not (ℓ, k)-consistent for any fixed ℓ, k.

Theorem (Feder, Vardi '98)

CSP is solvable by local consistency if and only if it does not have the "ability to count".

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Tractability of CSP

- (ℓ, k)-consistent CSPs are solvable in $O(|V|^k)$ time.
- Gaussian elimination runs in $O(|C|^3)$ time.
- Which CSPs are solvable in polynomial time?
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- Instead, assume P$\neq$NP, and prove NP-completeness.

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Computational Complexity

A computational problem is

- in P if finding a solution takes polynomial time.
- in NP if verifying a solution takes polynomial time.

Technical (but important) note: in the worst case.

- P $\subseteq$ NP. Whether the inclusion is strict is unknown.
- NP-complete problems are "the most difficult" in NP. Solving one in polynomial time means P=NP.
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Tractability of CSP

- Which CSPs are solvable in polynomial time?
- Don't know how to answer such questions unconditionally.
- Instead, assume P≠NP, and prove NP-completeness.
- CSP($\{0,1\}; \neq$) is in P.
- CSP($\{0,1\}; x \oplus y \oplus z = 0, x \oplus y \oplus z = 1$) is in P.
- CSP($\{0,1\}; x + y + z = 1$) is NP-complete.
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Any CSP that cannot "simulate" CSP($\{0,1\}; x + y + z = 1$) *is solvable in polynomial time.*

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Dichotomy Conjecture

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Any CSP with finite domain is either in P or NP-complete.

- If P$\subsetneq$NP, there are many artificial problems in between.
- In this sense, CSP is a natural computational problem.
- Proved in 2017 by Bulatov and Zhuk (independently).
- Both proofs use *universal algebra*.
- Intuitively, CSP is in P if allowed constraints share "symmetries", and NP-complete otherwise.

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Coping with NP-completeness

- Many interesting CSPs are NP-complete. What to do?
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  - ▷ Look for approximate solutions (MaxCSP).
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Infinite Domain CSPs

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